

# Balanced Topological Field Theories

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We describe a class of topological field theories called “balanced topological field theories.” These theories are associated to moduli problems with vanishing virtual dimension and calculate the Euler character of various moduli spaces. We show that these theories are closely related to the geometry and equivariant cohomology of “iterated superspaces” that carry two differentials. We find the most general action for these theories, which turns out to define Morse theory on field space. We illustrate the constructions with numerous examples. Finally, we relate these theories to topological sigma-models twisted using an isometry of the target space.

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## 1. Introduction and Conclusion

In recent years several examples of topological quantum field theories that compute the Euler number of particular moduli spaces have been investigated. For example, in [1][2] it was shown that the large  $N$  expansion of two-dimensional Yang-Mills theory [3] has a natural interpretation in terms of holomorphic maps from one Riemann surface to another. In order to write down a world-sheet action for this string theory topological field theories were constructed that calculate the Euler characters of moduli spaces of holomorphic maps. A very similar construction was employed in [4] to explain the occurrence of Euler characters of the moduli space of anti-selfdual connections in the topologically twisted  $N = 4$  supersymmetric Yang-Mills theory. The purpose of this paper is to clarify the underlying geometry of the constructions of [1][2][4] and to generalize these to a class of topological field theories we call “balanced topological field theories” (BTFT’s).

These models have the characteristic property of possessing two topological charges  $d_{\pm}$  and could very well be called  $N_T = 2$  topological field theories, where  $N_T$  is the number of *topological* charges. However, this might perhaps be confusing terminology, since topological field theories with one topological charge ( $N_T = 1$ ) are typically obtained by twisting supersymmetric field theories with two supercharges ( $N = 2$ ). Because of this possible confusion and because the ghost numbers are perfectly matched in these models, we prefer to use the term balanced field theories.

It turns out that these theories are intimately connected with a class of superspaces we call iterated superspaces. These spaces carry two exterior differentials  $d_{\pm}$ . We will show how the equivariant cohomology of these superspaces leads naturally to the peculiar field multiplets appearing in [1][2][4]. Moreover, these theories have a fundamental  $sl_2$  symmetry acting on the field space. The two BRST symmetries  $d_{\pm}$  transform covariantly under this  $sl_2$  symmetry. Since the formalism of topological field theory is very closely tied to de Rham cohomology, fiber bundles and equivariant cohomology, we will in this paper develop the generalizations of these concepts to the extended case.

One of the simplifying properties of balanced field theories is that the action can be determined from an *action potential*  $\mathcal{F}$ :

$$S = d_+ d_- \mathcal{F}. \tag{1.1}$$

As we will explain,  $\mathcal{F}$  should be thought of as a Morse function on field space. The path integral localizes to the critical points of  $\mathcal{F}$ . In our formalism gauging a symmetry is a

trivial operation: One simply uses the differentials of equivariant cohomology in (1.1). This aspect is but one of the various simplifying properties of  $N_T = 2$  topological field theories.

In this paper we focus on the geometrical foundations of the theory, and just briefly indicate the various applications. The outline of the paper is as follows. In section 2 we discuss the geometry of balanced topological theories in terms of iterated superspaces. We pay particular attention to the case where the underlying bosonic manifold is the total space of a vector bundle. Also, familiar concepts such as the Lie derivative and the inner product are given their appropriate generalizations. In section 3 we treat the equivariant case. For extended topological symmetry the geometry of principal bundles becomes very rich. In particular we will see that the curvature gets replaced by a full multiplet, consisting of a triplet of bosonic 2-form curvatures together with a doublet of fermionic 3-form curvatures. In section 4 we formulate balanced topological theories using the geometrical formalism developed in sections 2 and 3. We prove the existence of an action potential and make contact with the co-field formalism of [1][2]. In section 5 we prove the localization properties of a BTFT and show that it computes the Euler number. Also a very elegant formalism of gauge fixing is mentioned. Section 6 points out various examples, but they are not all treated in depth. Finally, section 7 contains a discussion of the relation with topological sigma-models, using a so-called isometry twist.

Finally we would like to point out that some aspects of our construction relating topological  $N_T = 2$  theories, Morse theory, the Matthai-Quillen formalism and Euler numbers of moduli spaces have also been investigated in [5][6] in a somewhat complementary fashion.

## 2. Geometry of Iterated Superspaces

In this section we collect various mathematical facts about the geometry of superspaces relevant to balanced topological field theories. This can be seen as a generalization of the usual apparatus of differential geometry, fiber bundles and cohomology to the case of more than one differential. It will give a natural interpretation of some of the results of [4]. Although one can easily discuss the general  $N_T$ -extended case, we restrict ourselves mainly to  $N_T = 2$  in this section.

### 2.1. The superspace $\widehat{X}$

We start with some very basic concepts. Let us first recall that to an  $n$  dimensional bosonic manifold  $X$  we can associate in a canonical way a  $(n|n)$  dimensional supermanifold  $\widehat{X} = \Pi TX$ . This supermanifold is modeled on the tangent bundle  $TX$  where the parity reversion operator  $\Pi$  acts by making the fibers anti-commuting. Any superspace is defined by its sheaf of functions. In the case of  $\widehat{X}$  this sheaf is generated by even and odd coordinates  $u^i, \psi^i$ , where we can think of  $\psi^i$  as the basis of one-forms  $du^i$ . So, over an open subset  $U \subset X$  the sheaf  $\mathcal{C}^\infty(\widehat{U})$  is given by the differential graded algebra (DGA)

$$\Lambda^*[\psi^i] \otimes \mathcal{C}^\infty(U), \quad (2.1)$$

with  $\Lambda^*$  the exterior algebra. Analysis on the supermanifold  $\widehat{X}$  is equivalent to studying differential forms on  $X$ , *i.e.* we can identify  $\mathcal{C}^\infty(\widehat{X}) \cong \Omega^*(X)$ . In this way the exterior differential  $d$  is represented by the odd vector field

$$d = \psi^i \frac{\partial}{\partial u^i}. \quad (2.2)$$

We refer to this well-known identification that underlies much of the applications of quantum field theory to topology as the “supertautology.”

Before we generalize this construction to more than one differential, we have to clarify one point. In the supergeometry of topological field theory one often considers the differential geometry on the total space of a vector bundle  $E \rightarrow B$ . In this case we would like to divide up the coordinates  $u^i$  into two sets: “basic coordinates”  $u^\mu$  and “fiber coordinates”  $\widehat{u}^a$ . Similarly, the anticommuting variables split into  $\psi^\mu = du^\mu$  and  $\widehat{\psi}^a = d\widehat{u}^a$ . The structure group of the sheaf of functions on  $\widehat{E}$  can be reduced and it is usually convenient to use the extra data of a connection  $\nabla$  on  $E$  to covariantize the action of the differential  $d$  on  $\widehat{E}$ . Thus, our sheaf of functions will be generated by variables  $(u^\mu, \psi^\mu; \widehat{u}^a, \widehat{\psi}^a)$  with  $(\psi^\mu; \widehat{u}^a, \widehat{\psi}^a)$  transforming linearly across patch boundaries on the base manifold  $B$ . We make this a sheaf of differential graded algebras (DGA’s) using the formula<sup>1</sup>

$$\begin{aligned} \nabla u &= \psi, \\ \nabla \psi &= \frac{1}{2} R \cdot u, \end{aligned} \quad (2.3)$$

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<sup>1</sup> Whenever we write expressions as  $R \cdot u$ , we assume that  $R$  is only contracted with the linear fiber coordinate  $\widehat{u}^a$  in  $u = (u^\mu; \widehat{u}^a)$ .

where  $R$  is a curvature. Here, and subsequently, we use the obvious identification of differential forms with polynomials in  $\psi^\mu$ , *i.e.* we write  $\nabla = d + \Gamma$ ,  $\Gamma = \Gamma_\mu \psi^\mu$ ,  $R = \frac{1}{2} R_{\mu\nu} \psi^\mu \psi^\nu$  etc., which act by linear transformations on the fiber variables. So, written out explicitly the above equations read

$$\begin{aligned} du^\mu &= \psi^\mu, \\ \nabla \hat{u}^a &\equiv d\hat{u}^a + \Gamma_{\mu b}^a \psi^\mu \hat{u}^b = \hat{\psi}^a, \\ d\psi^\mu &= 0, \\ \nabla \hat{\psi}^a &\equiv d\hat{\psi}^a + \Gamma_{\mu b}^a \psi^\mu \hat{\psi}^b = \frac{1}{2} R_{b\mu\nu}^a \psi^\mu \psi^\nu \hat{u}^b, \end{aligned} \tag{2.4}$$

where  $\Gamma_{\mu b}^a(u)$  is the local expression for the connection. The second line of (2.3) follows of course from consistency:  $\nabla^2 = R$ . The equations (2.4) should be regarded as *defining* the exterior derivative  $d$  on  $\hat{E}$ . Summarizing, we learn that in the case that  $X$  is the total space of a vector bundle, the fiber variable  $\hat{\psi}^a$  should be considered as the covariant differential  $\nabla \hat{u}^a$ .

## 2.2. The iterated superspace $\hat{\hat{X}}$

We now turn to the iterated superspace of  $X$ ,

$$\hat{\hat{X}} \equiv \Pi T(\Pi T X), \tag{2.5}$$

obtained by repeating the operation of the previous section once more. It is also defined by its functions  $F(u^i, \psi_A^i, H^i)$ . These now depend on two bosonic variables  $u^i, H^i$  and two fermionic variables  $\psi_A^i$ ,  $A = \pm$ . That is,  $\mathcal{C}^\infty(\hat{\hat{X}})$  is the sheaf on  $X$  which on an open set  $U \subset X$  is the algebra

$$\bigwedge^*[\psi_+^i, \psi_-^i] \otimes S^*[H^i] \otimes \mathcal{C}^\infty(U), \tag{2.6}$$

with  $S^*$  the symmetric algebra. Heuristically we think of the variables  $\psi_\pm^i$  as the one-forms  $d_\pm x^i$  obtained from two differentials  $d_\pm$ . The element  $H^i$  can then be thought of as  $d_- d_+ x^i = -d_+ d_- x^i$ . There are however some subtleties with a global interpretation along these lines as we discuss in a moment.

We obtained  $\hat{\hat{X}}$  by twice applying the operation  $\Pi T$ , but this obscures the natural action on the algebra of functions of the Lie algebra  $sl_2$  with generators  $J_{AB} = J_{BA}$  given

by

$$\begin{aligned}
J_{++} &= \psi_+^i \frac{\partial}{\partial \psi_-^i}, \\
J_{+-} &= J_{-+} = \psi_+^i \frac{\partial}{\partial \psi_+^i} - \psi_-^i \frac{\partial}{\partial \psi_-^i}, \\
J_{--} &= \psi_-^i \frac{\partial}{\partial \psi_+^i}.
\end{aligned} \tag{2.7}$$

The operator  $J_{+-}$  is the ghost number operator that counts the number of  $\psi_+$ 's minus the number of  $\psi_-$ 's. Under the algebra  $sl_2$  the fermions  $\psi_A$  form a doublet representation,  $u$  is a singlet and  $H$  a pseudo-singlet. Here “pseudo” means odd under charge conjugation  $+ \leftrightarrow -$ . That is, the combination  $\epsilon_{AB}H$  is invariant<sup>2</sup>. The operators  $d_A, J_{AB}$  form a closed algebra:  $\{d_A, d_B\} = 0$  and  $d_A$  is a doublet under the  $sl_2$  action.

Note that we can take an intermediate point of view and may consider (2.6) as defining the ring of differential forms  $\Omega^*(\widehat{X})$  on the superspace  $\widehat{X}$ . To do that we must break the  $sl_2$  symmetry and consider either  $(\psi_+^i, H^i)$  or  $(\psi_-^i, H^i)$  as one-forms.

As mentioned, we would like to turn  $\mathcal{C}^\infty(\widehat{X})$  into a BDGA (bi-differential graded algebra) with differentials  $d_\pm$ . How we do this depends on how we identify the algebras (2.6) across patches. One approach is to identify  $H^i = -d_+ \psi_-^i = d_- \psi_+^i$ . This gives a simple representation for the differentials  $d_A$  as

$$d_A = \psi_A^i \frac{\partial}{\partial u^i} + \epsilon_{AB} H^i \frac{\partial}{\partial \psi_B^i}, \tag{2.8}$$

but has the awkward feature that the variable  $H^i$  does not transform as a tensor but becomes a 2-jet with transformation rules

$$H^{i'} = \frac{\partial u^{i'}}{\partial u^j} H^j + \frac{\partial^2 u^{i'}}{\partial u^j \partial u^k} \psi_+^j \psi_-^k. \tag{2.9}$$

It is usually inconvenient to work with 2-jets, so we would rather define the sheaf of functions on  $\widehat{\widehat{X}}$  by the transformation rules:

$$\begin{aligned}
\psi_A^{i'} &= \frac{\partial u^{i'}}{\partial u^j} \psi_A^j, \\
H^{i'} &= \frac{\partial u^{i'}}{\partial u^j} H^j.
\end{aligned} \tag{2.10}$$

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<sup>2</sup> We use the convention that  $\epsilon^{+-} = -\epsilon_{+-} = 1$ .

According to the discussion of section 2.1 it is then necessary to introduce a connection  $\nabla$  on the tangent bundle  $TX$  and define the exterior differentials  $d_A$  in terms of this connection by the relations

$$\nabla_A \psi_B = \epsilon_{AB} H, \quad (2.11)$$

or, in full detail,

$$d_A \psi_B^i + \Gamma_{jk}^i \psi_A^j \psi_B^k = \epsilon_{AB} H^i. \quad (2.12)$$

We will adopt this second point of view in the development below. It has the consequence that the transformation of  $H$  is more complicated under the action of the differentials  $d_A$ :

$$\nabla_A H = -R_{AB} \psi_C \epsilon^{BC}. \quad (2.13)$$

The generators  $U = (u, \psi_A, H)$  of  $\mathcal{C}^\infty(\widehat{\widehat{X}})$  form what we will call a *basic quartet*. They can be arranged as

$$\begin{array}{ccccc} & & \psi_+^i & & 1 \\ & d_+ \nearrow & & \searrow d_- & \\ u^i & & & & H^i \\ & d_- \searrow & & \nearrow d_+ & \\ & & \psi_-^i & & -1 \end{array} \quad (2.14)$$

where we have indicated the action of the differentials and the ghost charges. We also note as in [4] that the generators can be conveniently combined into a  $N_T = 2$  superfield, by adding two odd variables  $\theta^+, \theta^-$ ,

$$U^i(\theta^+, \theta^-) = u^i + \theta^A \psi_A^i + \frac{1}{2} \epsilon_{AB} \theta^A \theta^B H^i. \quad (2.15)$$

### 2.3. Vector bundles

As we already mentioned, in many applications to topological field theories our space  $X$  actually will be the total space of a vector bundle  $E \rightarrow B$ . We will then have to consider the iterated superspace  $\widehat{\widehat{E}}$ . Thus, as in section 2.2, we will divide our coordinates on  $E$  again into fiber coordinates and base coordinates  $u^i = (u^\mu; \widehat{u}^a)$ . The functions on  $\widehat{\widehat{E}}$  are now generated by

$$U^i = (U^\mu; \widehat{U}^a) = (u^\mu, \psi_A^\mu, H^\mu; \widehat{u}^a, \widehat{\psi}_A^a, \widehat{H}^a), \quad (2.16)$$

with  $\psi_A^\mu, H^\mu; \widehat{u}^a, \widehat{\psi}_A^a, \widehat{H}^a$  transforming linearly across patch boundaries. In this case the differentials are defined by

$$\begin{aligned} \nabla_A u &= \psi_A, \\ \nabla_A \psi_B &= R_{AB} \cdot u + \epsilon_{AB} H, \\ \nabla_A H &= -R_{AB} \psi_C \epsilon^{BC} + P_A \cdot u, \end{aligned} \quad (2.17)$$

where the quantity  $P_A$  is a three-form and  $sl_2$  pseudo-doublet of bi-degrees  $(1, 2)$  and  $(2, 1)$ . It is defined by

$$P_A = \frac{1}{3} \nabla_B (R_{CA}) \epsilon^{BC}. \quad (2.18)$$

Its geometrical significance as a higher order form of curvature will become clear later. The appearance of terms of this nature is one of the new features of field theories with extended topological symmetries. For the moment we simply note that the objects  $R_{AB}, P_A$  satisfy

$$\begin{aligned} R_{AB} &= \frac{1}{2} [\nabla_A, \nabla_B], \\ P_A &= \frac{1}{6} [\nabla_B, [\nabla_C, \nabla_A]] \epsilon^{BC}. \end{aligned} \quad (2.19)$$

In all these formulas we use the obvious notation for identifying bi-graded differential forms with polynomials in the fermions  $\psi_A$ . That is, we have the identifications

$$\begin{aligned} R_{AB} &= R_{\mu\nu} \psi_A^\mu \psi_B^\nu, \\ P_A &= \frac{1}{3} \nabla_\mu R_{\nu\lambda} \psi_B^\mu \psi_C^\nu \psi_A^\lambda \epsilon^{BC} + \frac{2}{3} R_{\mu\nu} H^\mu \psi_A^\nu. \end{aligned} \quad (2.20)$$

Our notation is somewhat condensed. Thus, the first two lines of (2.17) are shorthand for:

$$\begin{aligned} d_A u^\mu &= \psi_A^\mu, \\ d_A \hat{u}^a + \Gamma_{\mu b}^a \psi_A^\mu \hat{u}^b &= \hat{\psi}_A^a, \\ d_A \psi_B^\mu + \Gamma_{\nu\lambda}^\mu \psi_A^\nu \psi_B^\lambda &= \epsilon_{AB} H^\mu, \\ d_A \hat{\psi}_B^a + \Gamma_{\mu b}^a \psi_A^\mu \hat{\psi}_B^b &= \frac{1}{2} R^a{}_{b\mu\nu} \psi_A^\mu \psi_B^\nu \hat{u}^b + \epsilon_{AB} \hat{H}^a \end{aligned} \quad (2.21)$$

and so on.

#### 2.4. De Rham cohomology of $\hat{X}$

Given a BDGA one can wonder what the properties of the corresponding cohomology theories are. A fundamental result for what follows is the following

**Theorem 2.1.** *Suppose  $\alpha$  is  $d_+$  and  $d_-$  closed and  $sl_2$  invariant, then  $\alpha$  can be decomposed as*

$$\begin{aligned} \alpha &= \alpha_0 + d_+ \beta_- + d_+ d_- \gamma \\ &= \alpha_0 - d_- \beta_+ + d_+ d_- \gamma \end{aligned} \quad (2.22)$$

where  $\alpha_0$  is constant on components of  $X$  and  $\beta_\pm = \beta_i \psi_\pm^i$ , with  $\beta_i du^i \in H^1(X)$ .



*Proof.* The proof of this theorem relies on constructing appropriate homotopy operators. For simplicity we will work in the case where the connection is zero (or  $H$  is a 2-jet). First we note that the algebra of functions on  $\widehat{\widehat{X}}$  is bigraded

$$\mathcal{C}^\infty(\widehat{\widehat{X}}) = \bigoplus_{q_+, q_- \geq 0} \mathcal{C}^{q_+, q_-}(\widehat{\widehat{X}}). \quad (2.23)$$

We define operators  $L_\pm$  measuring the separate charges  $q_\pm$

$$L_\pm = \psi_\pm^i \frac{\partial}{\partial \psi_\pm^i} + H^i \frac{\partial}{\partial H^i}. \quad (2.24)$$

We then introduce the homotopy operators

$$K_\pm = \pm \psi_\pm^i \frac{\partial}{\partial H^i}, \quad (2.25)$$

which satisfy the algebra

$$[d_\pm, K_\mp] = L_\mp. \quad (2.26)$$

This shows that as long as  $L$  is invertible, there is no cohomology. Thus we learn that the  $d_+$  cohomology is concentrated in degree  $(q_+, 0)$  while similarly  $d_-$  cohomology is concentrated in degree  $(0, q_-)$ . Now, assume  $\alpha$  is  $d_A$  closed and  $sl_2$  invariant and of degree  $(q_+, q_-)$ , which is not  $(0, 0)$  or  $(1, 1)$ . Then we may use the homotopy operators to show that  $\alpha$  is of the form  $\alpha = d_+ d_- \gamma$  in the following way:

$$\begin{aligned} \alpha &= \frac{1}{q_- (q_+ - 1)} d_+ d_- (K_+ K_- \alpha) \\ &= \frac{1}{q_+ (q_- - 1)} d_- d_+ (K_- K_+ \alpha). \end{aligned} \quad (2.27)$$

Similarly, if  $\alpha$  is of degree  $(q_+, q_-) = (1, 1)$  then

$$\begin{aligned} \alpha &= d_- (\psi_+^i A_i(U)) \\ &= -d_+ (\psi_-^i A_i(U)), \end{aligned} \quad (2.28)$$

where  $A_i du^i$  is a closed 1-form on  $X$ . ♠

## 2.5. Vector fields and derivations

Let  $V$  be a vector field on the bosonic manifold  $X$ .  $V$  induces two first order differential operators on  $\Omega^*(X)$ : the Lie derivative  $\mathcal{L}(V)$  and the contraction  $\iota(V)$  given by

$$\begin{aligned}\mathcal{L}(V) &= V^i \frac{\partial}{\partial u^i} + \frac{\partial V^i}{\partial u^j} \psi^j \frac{\partial}{\partial \psi^i}, \\ \iota(V) &= V^i \frac{\partial}{\partial \psi^i}.\end{aligned}\tag{2.29}$$

We can think of these derivations as vector fields on the superspace  $\widehat{X}$ . In particular,  $\mathcal{L}(V)$  can be interpreted as the lift  $\widehat{V}$  of the vector field  $V$  to  $\widehat{X}$ .

We can now repeat this procedure and lift the vector field (2.29) to a vector field  $\widehat{\widehat{V}}$  on the iterated superspace  $\widehat{\widehat{X}}$ :

$$\mathcal{L}(V) = \widehat{\widehat{V}} = V^i \frac{\partial}{\partial u^i} + \frac{\partial V^i}{\partial u^j} \psi_A^j \frac{\partial}{\partial \psi_A^i} + \left( \frac{\partial V^i}{\partial u^j} H^j - \frac{1}{2} \epsilon^{AB} \frac{\partial^2 V^i}{\partial u^k \partial u^j} \psi_A^j \psi_B^k \right) \frac{\partial}{\partial H^i}.\tag{2.30}$$

This derivation represents the Lie derivative on functions on  $\widehat{\widehat{X}}$ .

In a similar way we can represent the contraction of  $\widehat{V}$  on forms on  $\widehat{X}$  to define a doublet of contractions  $\iota^A(V)$  of bi-degree  $(-1, 0)$  and  $(0, -1)$

$$\iota^A(V) = V^i \frac{\partial}{\partial \psi_A^i} - \epsilon^{AB} \frac{\partial V^i}{\partial u^j} \psi_B^j \frac{\partial}{\partial H^i}.\tag{2.31}$$

Finally, in order to close the algebra of the operators  $d_A, \iota^A, \mathcal{L}$ , we have to introduce the operator  $I$ , a pseudo-scalar of bi-degree  $(-1, -1)$  defined as

$$I(V) = V^i \frac{\partial}{\partial H^i}.\tag{2.32}$$

By straightforward calculation one verifies that the above operators satisfy a bi-graded bi-differential Lie algebra based on  $Vect(X) \otimes \bigwedge^*[\eta_+, \eta_-]$ , where  $\eta_{\pm}$  are two anticommuting variables and the Lie bracket is given by  $[V \otimes \alpha, V \otimes \beta] = [V, W] \otimes \alpha\beta$ . One simply expands such a derivation in its even and odd components as

$$\mathcal{L}(V) + \eta_A \iota^A(V) + \frac{1}{2} \epsilon^{AB} \eta_A \eta_B I(V).\tag{2.33}$$

The nontrivial relations to verify are

$$\begin{aligned}
[d_A, \iota^B(V)] &= \delta_A^B \mathcal{L}(V), \\
[d_A, I(V)] &= -\epsilon_{AB} \iota^B(V), \\
[d_A, \mathcal{L}(V)] &= 0, \\
[\iota^A(V), \iota^B(W)] &= \epsilon^{AB} I([V, W]), \\
[\mathcal{L}(V), \iota^A(W)] &= \iota^A([V, W]), \\
[\mathcal{L}(V), I(W)] &= I([V, W]).
\end{aligned} \tag{2.34}$$

We will make use of the operators  $\iota^A, I$  when we consider the extended equivariant cohomology in the next section.

### 3. Extended Equivariant Cohomology

Much of the differential geometric framework of topological field theories is based on the concept of equivariant cohomology. Since we are interested in models with extended topological symmetry, we will develop in this section the notion of extended equivariant cohomology. We will meet some interesting generalizations of the notion of connection and curvature.

#### 3.1. The Weil and Cartan algebra

In the study of the differential geometry of a principal bundle  $P$  with Lie algebra  $\mathfrak{g}$  one encounters the so-called Weil algebra  $\mathcal{W}(\mathfrak{g})$ . Let us recall its definition; for more details see, for example, [2]. The Weil algebra is a DGA with  $\mathfrak{g}$ -valued generators  $\omega, \phi$  of degrees 1 and 2 respectively. The action of the differential  $d$  can be summarized in terms of the covariant derivative  $D = d + \omega$  by the relations

$$\phi = \frac{1}{2}[D, D], \quad [D, \phi] = 0. \tag{3.1}$$

The resulting action of  $d$  is

$$d\omega = \phi - \frac{1}{2}[\omega, \omega], \quad d\phi = -[\omega, \phi]. \tag{3.2}$$

These are of course the relations that are valid for a connection  $\omega$  and curvature  $\phi$  on a principal bundle  $P$ . In fact, one can define a connection simply as a homomorphism  $\mathcal{W}(\mathfrak{g}) \rightarrow \Omega^*(P)$ .

One interpretation of the relations (3.1) is that if we introduce the curvature  $\phi = D^2$  and take further commutators, then the process of introducing new generators stops because of the Bianchi-Jacobi identity  $[D, [D, D]] = 0$ .

One can introduce two derivations of the Weil algebra: the interior derivative or contraction

$$\iota_a \omega^b = \delta_a^b, \quad \iota_a \phi^b = 0, \quad (3.3)$$

(where  $\omega = \omega^a e_a, \phi = \phi^a e_a$ , with  $e_a$  a basis for  $\mathfrak{g}$ ) and the Lie derivative

$$\mathcal{L}_a = [\iota_a, d]. \quad (3.4)$$

The derivations  $d, \iota_a, \mathcal{L}_a$  satisfy a well known closed algebra.

The Cartan algebra  $\mathcal{C}(\mathfrak{g})$  is obtained by simply putting  $\omega = 0$  in the Weil algebra and is generated by the single variable  $\phi$  of degree two. We have the simple identity  $d\phi = 0$ , so the action of the differential  $d$  in the Cartan algebra is completely trivial.

### 3.2. Weil, BRST and Cartan models of equivariant cohomology

There are various ways to define the equivariant cohomology  $H_G^*(X)$  of a space  $X$  that carries the action of a (compact) Lie group  $G$ . Topologically it is defined as the cohomology of the space  $X_G = EG \times X/G$ , which is the universal  $X$ -bundle over the classifying space  $BG$ . However, there are also algebraic definitions. We briefly recall the so-called Weil, BRST and Cartan model.

For the Weil model we start with the algebra  $\mathcal{W}(\mathfrak{g}) \otimes \Omega^*(X)$ . On this algebra we have the action of the operators  $\iota_a$  and  $\mathcal{L}_a$ , now defined as  $\iota_a = \iota_a \otimes 1 + 1 \otimes \iota_a$  etc. Here we write  $\iota_a = \iota(V_a)$ , with  $V_a$  the vector field on  $X$  corresponding to the Lie algebra element  $e_a$ . One then restricts to the so-called basic forms  $\eta \in \mathcal{W}(\mathfrak{g}) \otimes \Omega^*(X)$  which satisfy  $\iota_a \eta = \mathcal{L}_a \eta = 0$ . The equivariant cohomology groups are then defined as

$$H_G^*(X) = H^*((\mathcal{W}(\mathfrak{g}) \otimes \Omega^*(X))_{basic}, d^W) \quad (3.5)$$

with Weil differential  $d^W = d \otimes 1 + 1 \otimes d$ .

The BRST model is simply related to the Weil model. One starts from the space basic forms, but uses the differential

$$d^{BRST} = d^W + \omega^a \otimes \mathcal{L}_a - \phi^a \otimes \iota_a. \quad (3.6)$$

It was shown in [7] that the two models are related as

$$d^{BRST} = e^{\omega^a \iota_a} d^W e^{-\omega^a \iota_a}. \quad (3.7)$$

There is a simpler model of equivariant cohomology — the Cartan model, based on the Cartan algebra  $\mathcal{C}(\mathfrak{g}) = S^*(\mathfrak{g}^*)$ . The starting point is now the algebra  $S^*(\mathfrak{g}^*) \otimes \Omega^*(X)$ , but as differential we choose

$$d^C = 1 \otimes d - \phi^a \otimes \iota_a. \quad (3.8)$$

This operator satisfies  $(d^C)^2 = -\phi^a \otimes \mathcal{L}_a$  and thus only defines a complex on the  $G$ -invariant forms. The Cartan model of equivariant cohomology is now defined as

$$H_G^*(X) = H^*((S^*(\mathfrak{g}^*) \otimes \Omega^*(X))^G, d^C). \quad (3.9)$$

One can then show that the definitions (3.5) and (3.9) are equivalent and agree with the topological definition.

### 3.3. Algebra of $N$ -extended covariant derivatives

If we have a geometry such as the iterated superspaces  $\widehat{\widehat{X}}$  where it is natural to introduce several independent exterior derivatives, then the principal bundles over such spaces will also have several covariant derivatives. This motivates the following generalization of the Weil algebra which we call the Weil algebra of order  $N$  and denote as  $\mathcal{W}^N(\mathfrak{g})$ .

We introduce several covariant derivatives and connections

$$D_A = d_A + \omega_A, \quad A = 1, \dots, N, \quad (3.10)$$

and then introduce successive “curvatures”

$$\phi_{A_1 \dots A_j} = \frac{1}{j!} [D_{A_1}, [D_{A_2}, \dots, D_{A_j}] \dots] \quad (3.11)$$

until the Bianchi identities close the algebra. The resulting DGA is the Weil algebra of order  $N$ . One can define various Cartan models by putting generators to zero. We will discuss the case  $N = 2$  in detail in the next subsection. Here we restrict ourselves to a general description of the Cartan algebra  $\mathcal{C}^N(\mathfrak{g})$  of order  $N$ .

This extended Cartan algebra can be abstractly described as follows. Let  $V$  be the  $N$  dimensional *odd* vector space generated by the basis elements  $D_A$  of degree 1. Let  $L = \text{Free}(V)$  be the *free* Lie algebra on  $V$ . This is the space spanned by all possible

commutators of the  $D_A$ 's imposing the relations following from the (anti)symmetry of the Lie bracket and the Jacobi identity. So at degree two we have the elements  $[D_A, D_B] = [D_B, D_A]$  etc. Now let  $L'$  denote the subalgebra of the elements of  $L$  with degree  $\geq 2$ . We easily see that as a Lie algebra  $L'$  is generated by the elements of  $V' \equiv L'/[L', L']$ , that is,  $L' = \text{Free}(V')$ . One can show that  $V'$  is finite dimensional and concentrated in degrees  $2, \dots, N$ . We can think of  $V'$  as the space of curvatures

$$\phi_{AB} = \frac{1}{2}[D_A, D_B], \quad (3.12)$$

and higher order generalizations.

The elements  $D_A$  in  $V$  (of degree 1) act as differentials (denoted as  $d_A$ ) on the vector space  $V'$  by

$$d_A \eta = [D_A, \eta], \quad (3.13)$$

and commute  $[d_A, d_B] = 0$  within  $V'$ . Actually, the differentials  $d_A$  will only commute up to commutator terms if we pick explicit representatives of  $V'$  in  $L'$ . In fact, if we pick a basis  $\phi_I$  of  $V'$  ( $I$  will be a multi-index in terms of the indices  $A, B, \dots = 1, \dots, N$ .) and keep the same notation for its representatives in  $L'$ , we have an action of  $d_A$  of the form

$$d_A \phi_I = c_{AI}{}^J \phi_J + c_{AI}{}^{JK} [\phi_J, \phi_K] + \dots \quad (3.14)$$

where the ellipses indicate higher order commutators. Now within  $L'$  the differentials satisfy

$$[d_A, d_B] = 2[\phi_{AB}, \cdot]. \quad (3.15)$$

We now define the Cartan algebra  $\mathcal{C}^N(\mathfrak{g})$  as the DGA  $S^*(\mathfrak{g}^* \otimes V')$ , where we define the differential  $d_A$  by evaluating all the Lie brackets in (3.14) in  $\mathfrak{g}$ . That is,  $\mathcal{C}^N(\mathfrak{g})$  is the algebra generated by the  $\phi_I$  which now take values in  $\mathfrak{g}$ . We will see in the next section what this all means concretely for  $N = 2$ . Note that in contrast with the case  $N = 1$ , for the case  $N > 1$  we still have a nontrivial set of differentials  $d_A$  acting on the Cartan algebra.

### 3.4. $N = 2$ Weil and Cartan algebra

We now focus on the case  $N = 2$ , as is appropriate to  $\widehat{\widehat{X}}$ . The  $N = 2$  Weil model  $\mathcal{W}^2(\mathfrak{g})$  is the unique BDGA with generators (see also [8] for a some what different definition of a bigraded version of the Weil algebra)

$$\begin{array}{ccccc}
 & & & \phi_{++} & \\
 & \omega_+ & & & \eta_+ \\
 \text{connections:} & & \Omega & \text{curvatures:} & \phi_{+-} \\
 & \omega_- & & & \eta_- \\
 & & & \phi_{--} & 
 \end{array} \tag{3.16}$$

Here we indicated the ghost charges graphically. The generators  $\omega_A, \phi_{AB}, \Omega, \eta_A$  have the following degrees and  $sl_2$  representations  $(1, \mathbf{2}), (2, \mathbf{3}), (2, \mathbf{1}'), (3, \mathbf{2}')$ . (The primes indicate pseudo-representations.) They satisfy the relations

$$\begin{aligned}
 \Omega &= d_+ \omega_- - d_- \omega_+, \\
 \phi_{AB} &= \frac{1}{2} [D_A, D_B], \\
 \eta_A &= -\frac{1}{6} [D_B, [D_C, D_A]] \epsilon^{BC}.
 \end{aligned} \tag{3.17}$$

We can define the Cartan model by putting  $\omega_A, \Omega = 0$ . This leaves us just the variables  $\phi_{AB}, \eta_A$  in (3.16). The transformation laws become now

$$\begin{aligned}
 d_A \phi_{BC} &= \epsilon_{AB} \eta_C + \epsilon_{AC} \eta_B, \\
 d_A \eta_B &= -\frac{1}{2} [\phi_{AC}, \phi_{BD}] \epsilon^{CD},
 \end{aligned} \tag{3.18}$$

reproducing the transformation laws of [4].

Let us make a comment about the object  $\eta_A$ , because it illustrates very well the new features of the extended algebras. Indeed, let us see why these objects appear according to the general definition given in the previous subsection. In the  $N = 2$  case we have two covariant derivatives  $D_A$  in degree one and three curvatures  $\phi_{AB} = \frac{1}{2} [D_A, D_B]$  in degree two. In degree three we have the triple commutators  $[D_A, [D_B, D_C]]$ . However, for  $N = 2$  the six independent triple commutators  $[D_A, [D_B, D_C]]$  are not all determined by the Jacobi identity. This should be compared to the  $N = 1$  case, where the Jacobi-Bianchi identity gives us  $[D, [D, D]] = [D, \phi] = 0$ . In fact, for  $N = 2$  there are only four Jacobi-Bianchi identities which are given by

$$\begin{aligned}
 [D_+, \phi_{++}] &= 0, & 2[D_+, \phi_{+-}] + [D_-, \phi_{++}] &= 0, \\
 [D_-, \phi_{--}] &= 0, & 2[D_-, \phi_{+-}] + [D_+, \phi_{--}] &= 0.
 \end{aligned} \tag{3.19}$$

This implies that there are two (six minus four) new generators  $\eta_A$  at degree three. Equation (3.17) implies that these are explicitly given by

$$\eta_+ = -[D_+, \phi_{+-}], \quad \eta_- = [D_-, \phi_{+-}]. \quad (3.20)$$

One easily verifies that at degree four and higher no new generators appear. So we learn that

$$\mathcal{C}^2(\mathfrak{g}) = S^*(\phi_{AB}^a, \eta_A^a), \quad A, B = \pm, \quad a = 1, \dots, \dim \mathfrak{g}. \quad (3.21)$$

### 3.5. $N = 2$ extended equivariant cohomology

We are now in a position to discuss the equivariant cohomology of iterated superspaces. Suppose  $X$  has a  $G$  action generated by vector fields  $V_a$  where  $e_a$  denote a basis of the Lie algebra  $\mathfrak{g}$  of  $G$ . By lifting these vector fields as described in section 2.5, we obtain a  $G$  action on the space  $\widehat{\widehat{X}}$ , together with the derivations  $d_A, \mathcal{L}(V_a), \iota_A(V_a), I(V_a)$  of  $\mathcal{C}^\infty(\widehat{\widehat{X}})$ .

As in the case  $N = 1$ , there are several models for the equivariant cohomology, We discuss here briefly the Weil model, BRST model and Cartan model. For the Weil model we consider the complex  $\mathcal{W}^2(\mathfrak{g}) \otimes \mathcal{C}^\infty(\widehat{\widehat{X}})$  and the differential is simply the sum of the two differentials as defined above:

$$d_A^W = d_A \otimes 1 + 1 \otimes d_A, \quad (3.22)$$

acting on the basic forms, that are now defined to satisfy

$$\iota_A \alpha = \mathcal{L} \alpha = I \alpha = 0. \quad (3.23)$$

The BRST model is defined analogously as in (3.6)

$$d_A^{BRST} \equiv \exp \left[ \iota^A(\omega_A) + I(\Omega) \right] d_A^W \exp \left[ -\iota^A(\omega_A) - I(\Omega) \right] \quad (3.24)$$

Finally, the Cartan model is based on the  $G$ -invariant subalgebra of  $\mathcal{C}^2(\mathfrak{g}) \otimes \mathcal{C}^\infty(\widehat{\widehat{X}})$  with equivariant differential

$$d_A^C = d_A^W + \phi_{AB}^a \iota^B(V_a) + \eta_A^a I(V_a). \quad (3.25)$$



This gives the explicit transformation laws (3.18) together with the following action of the Cartan differential on functions on  $\widehat{\widehat{X}}$  (or, equivalently, differential forms on  $\widehat{X}$ )

$$\begin{aligned} d_A u^i &= \psi_A^i \\ d_A \psi_B &= \mathcal{L}(\phi_{AB})u + \epsilon_{AB}H \\ d_A H &= -\mathcal{L}(\phi_{AB})\psi_C \epsilon^{BC} - \mathcal{L}(\eta_A) \cdot u \end{aligned} \tag{3.26}$$

where we have dropped the superscript on  $d_A$  and used the compressed notation  $\phi_{AB} = \phi_{AB}^a V_a$  etc. This again reproduces the transformation laws in [4].

The extended equivariant cohomology is not very different from the ordinary equivariant cohomology. One can show that the  $N = 2$  equivariant cohomology of  $X$  is actually isomorphic to that of  $X$ , at least outside of degrees  $(a, b)$  for  $a, b = 0, 1$ . To prove this we introduce the homotopy operator  $K = K^X + K^C$  where  $K^C$  for the Cartan model is defined by  $K^C \eta_+ = -\phi_{+-}, K^C \eta_- = -\frac{1}{2}\phi_{--}$  with  $K = 0$  on all other generators and  $K^X = K_-$  is defined in (2.25). A short calculation shows that

$$[d_+, K] = L_- + \eta_A^a \frac{\partial}{\partial \eta_A^a} + \phi_{+-}^a \frac{\partial}{\partial \phi_{+-}^a} + \phi_{--}^a \frac{\partial}{\partial \phi_{--}^a} \tag{3.27}$$

from which the result follows.

#### 4. Balanced Topological Field Theory

We now introduce a new class of topological field theories, which include the “cofield construction” of [1][2] as a special case. One natural name for these theories would be  $N_T = 2$  topological field theories. Here  $N_T$  denotes the number of *topological* supercharges or BRST operators. This should not be confused with extended supersymmetric theories. In fact, the twisting procedure will typically relate models with  $N = 2$  supersymmetry to  $N_T = 1$  topological symmetry and models with  $N = 4$  supersymmetry to  $N_T = 2$  topological field theories. Since this nomenclature has perhaps too many misleading connotations and since the ghosts and antighosts are perfectly matched in these theories we propose to call them “balanced topological field theories” (BTFT’s).

#### 4.1. Review of the standard construction of TFT

The basic data for a TFT are (i) a space  $\mathcal{C}$  of fields, (ii) a bundle  $E \rightarrow \mathcal{C}$  of equations equipped with a metric  $(\cdot, \cdot)_E$  and connection  $\nabla$  compatible with the metric, and (iii) a section  $s \in \Gamma(E)$  such that its zero locus  $\mathcal{M} = \mathcal{Z}(s)$  defines a moduli problem of interest. (This is reviewed in detail in [2], see *e.g.* also [9].)

The construction of the topological field theory can be phrased in terms of the supergeometry of  $\widehat{E}^*$ . As in section 2.1 we wish to distinguish the fiber coordinates from the field space coordinates  $u^\mu, \psi^\mu$ . The fiber coordinates are the “antighosts,” coordinates on the dual to the bundle of equations:

$$\begin{aligned}\rho_a &\in \Omega^0(M; \Pi E^*), \\ H_a &\in \Omega^1(M; \Pi E^*).\end{aligned}\tag{4.1}$$

The bundle  $E^*$  carries a connection  $\nabla$  with curvature  $R$  and the BRST operator  $Q = d$  is defined by:

$$\begin{aligned}\nabla \rho &= H, \\ \nabla H &= R \cdot \rho.\end{aligned}\tag{4.2}$$

The topological field theory action  $I$  is defined in terms of the gauge fermion

$$\Psi = i\langle \rho, s \rangle - (\rho, \nabla \rho)_{E^*}\tag{4.3}$$

in the form

$$I = Q\Psi = iH_a s^a - (H, H) - i\langle \rho, \nabla s \rangle + (\rho, R\rho)_{E^*}\tag{4.4}$$

where we use the compatibility of the metric and connection. General arguments show that the path integral  $Z = \int e^{-I}$  computes the Euler character of the bundle of antighost zero modes over the moduli space  $\mathcal{Z}(s)$ :

$$Z = \int_{\mathcal{Z}(s)} \chi(\text{cok } \nabla s).\tag{4.5}$$

The above story becomes a little more intricate in the presence of a gauge symmetry  $G$ . The basic topological multiplet  $(A, B)$  takes values in an equivariant bundle over field space with connection  $\nabla$  and has transformation laws:

$$\begin{aligned}\nabla A &= B, \\ \nabla B &= \mathcal{R} \cdot A,\end{aligned}\tag{4.6}$$

where the combination  $\mathcal{R} = R + \mathcal{L}(\phi)$  is the equivariant curvature [10].

In order to construct the Poincaré dual to the moduli space  $\mathcal{Z}(s)/G$  one introduces the extra multiplet  $\lambda, \eta \in \mathfrak{g} = \text{Lie}(G)$  of degree  $-2, -1$ . We will write here the Lie algebra indices as  $\lambda^x, \eta^x$ . Let us denote the vertical vector fields associated with the gauge group action by:

$$\begin{aligned} (\mathcal{L}(\lambda)u)^I &= \lambda^x V_x^I(u), \\ V_x^I(u) &= C : \mathfrak{g} \rightarrow T_u \mathcal{C} \end{aligned} \tag{4.7}$$

and define the projection gauge fermion:  $\Psi_{proj} = i(\psi, \mathcal{L}(\lambda) \cdot u)$  to project out the redundant gauge degrees of freedom. The resulting term in the action is:

$$Q\Psi_{proj} = (\lambda, C^\dagger C\phi + C^\dagger Ru + \partial_J(C^\dagger)^x_I \psi^I \psi^J) - (\psi, \mathcal{L}(\eta)u). \tag{4.8}$$

Note that  $\lambda$  is a Lagrange multiplier and the resulting delta function fixes  $\phi$  away from fixed points of the gauge group.

The fermion kinetic terms may be written as:

$$i\rho_a \nabla_I s^a \psi^I + i\eta^x (C^\dagger)^x_I \psi^I = (\rho \quad \eta) \mathbb{O} \psi, \tag{4.9}$$

where the operator  $\mathbb{O}$  is defined by:

$$T\mathcal{C} \xrightarrow{\mathbb{O} = \nabla s \oplus C^\dagger} \Omega^1(\mathcal{C}; E) \oplus \mathfrak{g}^*, \tag{4.10}$$

and is associated to the deformation complex

$$0 \rightarrow \mathfrak{g} \xrightarrow{C} T\mathcal{C} \xrightarrow{\nabla s} E \rightarrow 0 \tag{4.11}$$

by using the metric. (4.11) is a complex if the equations are gauge invariant. The complex is exact at degree  $-1$ , if the group action is free. Again general arguments show that the path integral is just:

$$Z = \int_{\mathcal{Z}(s)/G} \chi(\text{cok } \mathbb{O}/G). \tag{4.12}$$

#### 4.2. Balanced topological field theories: field content

In a balanced or  $N_T = 2$  topological field theory, the fields in the model are the generators of functions on  $\widehat{\widehat{X}}$ . We will denote coordinates on  $X$  by  $u^i$ . Sometimes we will divide up the coordinates into fiber and basic coordinates. As usual the generators form a quartet:

$$\begin{array}{ccccc}
 & & \psi_+^i & & \\
 & \nearrow & & \searrow & \\
 u^i & & & & H^i \\
 & \searrow & & \nearrow & \\
 & & \psi_-^i & & 
 \end{array} \tag{4.13}$$

where we note that all of  $\psi_A^i, H^i$  should be regarded as (even or odd) sections of a vector bundle. These bundles have connections so we can define the differentials as in (2.17). We will assume a group  $G$  acts on  $X$  and introduce the Cartan multiplet  $\phi_{AB}, \eta_A$  as in (3.16). The  $G$ -equivariant BRST differentials are now defined to act by

$$\begin{aligned}
 \nabla_A u &= \psi_A, \\
 \nabla_A \psi_B &= \mathcal{R}_{AB} \cdot u + \epsilon_{AB} H, \\
 \nabla_A H &= -\mathcal{R}_{AB} \psi_C \epsilon^{BC} + \mathcal{P}_A \cdot u,
 \end{aligned} \tag{4.14}$$

where the geometrical operators are defined by

$$\begin{aligned}
 \mathcal{R}_{AB} &= R_{AB} + \mathcal{L}(\phi_{AB}), \\
 \mathcal{P}_A &= \frac{1}{3} \nabla_B (\mathcal{R}_{CA}) \epsilon^{BC} = P_A + \mathcal{L}(\eta_A), \\
 \mathcal{P}_\pm &= \pm \nabla_\pm \mathcal{R}_{\pm\mp}.
 \end{aligned} \tag{4.15}$$

Here  $\mathcal{R}_{AB}$  and  $\mathcal{P}_A$  are the  $N_T = 2$  extended equivariant curvatures.

#### 4.3. Balanced topological field theories: The action potential

A topological field theory with field space of the form  $\widehat{\widehat{\mathcal{C}}}$  is called *balanced* if the action is an  $sl_2$  invariant and  $d_+, d_-$  closed function on  $\widehat{\widehat{\mathcal{C}}}$ . Let us characterize the most general action of a BTFT. The action  $I$ , being a function in  $\mathcal{C}(\widehat{\widehat{X}})$ , carries a bigrading  $(q_+, q_-)$ . According to Theorem 2.1 the action is both  $d_+$  and  $d_-$  exact and is, in fact, of the form:

$$I = I_0 + d_+ d_- \mathcal{F}. \tag{4.16}$$

where the “topological term”  $I_0(u)$  is constant on the components of  $\mathcal{C}$ . We refer to  $\mathcal{F}$  as the “action potential” since it is analogous to the Kähler potential of a Kähler form. Note that  $\mathcal{F}$  is not uniquely defined; we can always shift

$$\mathcal{F} \rightarrow \mathcal{F} + d_+ \Phi_- + d_- \Phi_+. \quad (4.17)$$

Note further that if  $H^1(\mathcal{C}) \neq 0$  then  $\mathcal{F}$  need not be globally well-defined on field space. So the analogy to a Kähler potential is quite good.

By  $sl_2$  invariance, action potentials must be of total ghost charge zero. The most natural action potentials are of the form

$$\mathcal{F} = \mathcal{F}_0(u) + \epsilon(\psi_+, \psi_-) + \beta(H, u) + \gamma(\eta_+, \eta_-), \quad (4.18)$$

where  $(\cdot, \cdot)$  is a metric on the bundles over field space which is compatible with the connections and  $\mathcal{F}_0(u)$  is a function on field space. Locally, this is the most general action potential which is at most first order in  $\psi_\pm, H, \eta_\pm$ .

Let us discuss the separate terms individually. The gauge fermions  $\Psi_- = d_- \mathcal{F}$  and actions  $S = d_+ \Psi_- = d_+ d_- \mathcal{F}$  associated with these terms are:

- $\mathcal{F}_0(u)$ . We will assume that  $\mathcal{F}_0(u)$  is a  $G$ -invariant function. Then:

$$\begin{aligned} \Psi_- &= \nabla_I \mathcal{F}_0 \psi_-^I, \\ d_+ d_- \mathcal{F}_0(u) &= -H^I \nabla_I \mathcal{F}_0 + \frac{1}{2} \epsilon^{AB} \psi_A^I \psi_B^J \nabla_J \nabla_I \mathcal{F}_0. \end{aligned} \quad (4.19)$$

- $(\psi_+, \psi_-)$ . The fermion bilinear gives rise to

$$\begin{aligned} \Psi_- &= (H, \psi_-) + (\mathcal{R}_{-+} u, \psi_-) - (\mathcal{R}_{--} u, \psi_+), \\ d_+ d_- (\psi_+, \psi_-) &= -(H, H) + 2(\mathcal{P}_A u, \psi_B) \epsilon^{AB} \\ &\quad - \frac{1}{2} \epsilon^{AC} \epsilon^{BD} \left[ (\mathcal{R}_{AB} u, \mathcal{R}_{CD} u) + 2(\psi_A, \mathcal{R}_{BC} \psi_D) \right] \end{aligned} \quad (4.20)$$

- $(H, u)$  is equivalent to  $(\psi_+, \psi_-)$ . This follows from the identity

$$(H, u) = (\psi_-, \psi_+) - d_- (\psi_+, u). \quad (4.21)$$

- $(\eta_+, \eta_-)$ . This equivariant term gives the following contributions to the gauge fermion and action

$$\begin{aligned} \Psi_- &= \frac{1}{2}([\phi_{--}, \phi_{++}], \eta_-) - (\eta_+, [\phi_{--}, \phi_{-+}]), \\ d_+ d_- (\eta_+, \eta_-) &= ([\phi_{++}, \phi_{+-}], [\phi_{--}, \phi_{-+}]) + ([\phi_{++}, \phi_{--}], [\phi_{++}, \phi_{--}]) \\ &\quad + \epsilon^{AB} \epsilon^{CD} ([\eta_A, \phi_{BC}], \eta_D). \end{aligned} \quad (4.22)$$

In section 5 below we will show that under good conditions the path integral for the theory (4.18) localizes to the critical submanifold of  $\mathcal{F}_0$  modulo gauge transformations:

$$\mathcal{M} = \{u : \nabla \mathcal{F}_0(u) = 0\} / G \quad (4.23)$$

and that, moreover, the partition function computes the Euler number of this moduli space,

$$Z = \chi(\mathcal{M}) \quad (4.24)$$

Thus, balanced topological field theories compute Morse theory on field space, with the action potential serving as a Morse function.

#### 4.4. Viewing BTFT as a standard TFT

The transformation laws (4.14) are not standard TFT transformations. But we may make the redefinition  $H' = \mathcal{R}_{+-}u - H$  and then view the theory as a standard one with the following familiar field content:

- Matter multiplets:

$$\begin{aligned} \nabla_+ u &= \psi_+, \\ \nabla_+ \psi_+ &= \mathcal{R}_{++}u, \\ \nabla_+ \phi_{+-} &= -\eta_+, \\ \nabla_+ \eta_+ &= -[\phi_{++}, \phi_{+-}]; \end{aligned} \quad (4.25)$$

- Antighosts:

$$\begin{aligned} \nabla_+ \psi_- &= H', \\ \nabla_+ H' &= \mathcal{R}_{++}\psi_-; \end{aligned} \quad (4.26)$$

- Projection multiplet:

$$\begin{aligned} \nabla_+ \phi_{--} &= -2\eta_-, \\ \nabla_+ \eta_- &= -\frac{1}{2}[\phi_{++}, \phi_{--}]; \end{aligned} \quad (4.27)$$

- Gauge fermion for equations

$$i\psi_-^I \nabla_I \mathcal{F}_0 + 2\alpha(\psi_-, \mathcal{L}(\phi_{+-})u) + \alpha(H', \psi_-); \quad (4.28)$$

- Projection gauge fermion

$$-\alpha(\psi_+, \mathcal{L}(\phi_{--})u) - \frac{1}{2}\gamma(\eta_+, [\phi_{--}, \phi_{+-}]). \quad (4.29)$$

The rest of the gauge fermion following from the action potential is then declared an irrelevant  $Q$ -exact modification.

#### 4.5. The cofield construction

The “co-field construction” described in [1][2][4] is a map by which we can assign a BTFT to any TFT. Under good conditions this will compute the Euler character of the original moduli space to which the TFT localizes.

We return to the original moduli problem in section 4.1 defined by the vanishing of a section  $s^a(u)$  in the “bundle of equations. The basic idea is to take  $X = E^*$  as the field space with coordinates  $U^I = (u^\mu; \hat{u}_a)$ . The degree 0 part of the action potential is then simply

$$\mathcal{F}_0(U) = \hat{u}_a s^a(u). \quad (4.30)$$

Clearly, the critical points of this Morse function are:

$$s^a(u) = 0, \quad \nabla_\mu s^a \hat{u}_a = 0. \quad (4.31)$$

If  $s$  is sufficiently nondegenerate the second equation implies  $\hat{u}_a = 0$  and the solutions to the equations is the same moduli space as in the original TFT. By (4.24) we see that the BTFT will calculate the Euler character of this moduli space.

It is straightforward to implement this idea in detail. The fields generate  $\widehat{\widehat{E^*}}$ . They may be arranged into two basic quartets:

$$\begin{array}{ccccc} & & \chi^\mu & & \\ & \nearrow & & \searrow & \\ u^\mu & & & & \hat{H}^\mu \\ & \searrow & & \nearrow & \\ & & \hat{\rho}^\mu & & \end{array} \quad (4.32)$$

filling out the “fields” of the original moduli problem, and

$$\begin{array}{ccccc} & & \hat{\chi}_a & & \\ & \nearrow & & \searrow & \\ \hat{u}_a & & & & H_a \\ & \searrow & & \nearrow & \\ & & \rho_a & & \end{array} \quad (4.33)$$

filling out the “antighosts” of the original moduli problem. The BTFT action potential is:

$$\begin{aligned} \mathcal{F} &= i\mathcal{F}_0(U) - (\psi_-^\mu \quad \psi_{-,a}) \begin{pmatrix} G_{\mu\nu} & G_\mu^b \\ G_\nu^a & G^{ab} \end{pmatrix} \begin{pmatrix} \psi_+^\nu \\ \psi_{+,b} \end{pmatrix} \\ \mathcal{F}_0(U) &= \hat{u}_a s^a(u), \end{aligned} \quad (4.34)$$

where  $G$  is a metric. The construction is easily “equivariantized” by using the equivariant differentials.

#### 4.6. Summary: the deformation complexes

The various classes of topological field theories are nicely summarized by their associated deformation complexes:

- For a general *topological field theory* we have the usual complex

$$0 \rightarrow \mathbf{g} \xrightarrow{C} TC \xrightarrow{\nabla_s} E \rightarrow 0 \quad (4.35)$$

of symmetries, fields, and equations [11].

- For a *balanced topological field theory* we get the complex

$$0 \rightarrow \mathbf{g} \xrightarrow{(C,0)} TC \oplus \mathbf{g} \xrightarrow{(\nabla^2 \mathcal{F}_0, C)} TC \rightarrow 0, \quad (4.36)$$

where the maps act as

$$\begin{aligned} \eta_- &\rightarrow (C\eta_-, 0) \\ (\psi_+, \eta_+) &\rightarrow \nabla^2 \mathcal{F}_0 \psi_+ + C\eta_+. \end{aligned} \quad (4.37)$$

- The *cofield construction* is associated with the complex <sup>3</sup>

$$0 \rightarrow \mathbf{g} \longrightarrow (E^* \oplus TC) \oplus \mathbf{g} \longrightarrow (E^* \oplus TC) \rightarrow 0, \quad (4.38)$$

where the maps are defined as

$$\begin{aligned} \eta_- &\rightarrow (C\eta_-, 0, 0) \\ (\hat{\psi}_+^a, \psi_+^i, \eta_+) &\rightarrow \nabla^2 \mathcal{F}_0 \psi_+ + C\eta_+. \end{aligned} \quad (4.39)$$

### 5. Localization of BTFT

In this section we justify the localization result (4.23) and (4.24) more fully. As we have seen, the general action potential can be taken to be a sum of a function  $\mathcal{F}_0$  on field space and quadratic terms in the fermions  $\psi_A$  and  $\eta_A$ :

$$\mathcal{F} = i\mathcal{F}_0(u) + \alpha(\psi_+, \psi_-) + \gamma(\eta_+, \eta_-), \quad (5.1)$$

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<sup>3</sup> The rolled-up complex of the cofield construction suggests a role for quaternionic vector spaces. Moreover, these equations suggest a duality between equations and symmetries. We thank Andrei Losev for an interesting discussion about this.



where  $\alpha, \gamma$  are constants. Putting  $\alpha$  to zero results in a singular Lagrangian and an ill-defined path integral. The coefficient  $\gamma$  is subtle and is related to the introduction of mass terms into topological field theory. The general discussion of the localization of the theory based on the action potential (5.1) is quite involved. We will simply illustrate it for the following situation:

(i) All the curvatures and connections can be set to zero. This is the case for topological Yang-Mills, where  $\mathcal{C}$  is an affine space and for 2D topological gravity in the Beltrami formulation.

(ii) The gauge group  $G$  acts without fixed points.

(iii) The coefficient  $\gamma = 0$ . (Otherwise the action is not quadratic in  $\phi_{++}, \phi_{--}$ .)

(iv) All the zero modes of the Hessian of  $\mathcal{F}_0$  on critical submanifolds are associated with gauge symmetries or tangent directions to the moduli space.

In the case that the conditions (i)–(iv) are satisfied, we can justify the localization to (4.23) above, as we will now demonstrate. Let us introduce the notation

$$(\psi_1, \mathcal{L}(\phi) \cdot \psi_2)_{T^*\mathcal{C}} \equiv (\phi, K(\psi_1, \psi_2))_{\mathbf{g}} \quad . \quad (5.2)$$

The action (5.1) becomes:

$$d_+ d_- \mathcal{F} = L_1 + L_2 + \alpha L_3 + \alpha L_4, \quad (5.3)$$

with

$$\begin{aligned} L_1 &= -i \langle \nabla \mathcal{F}_0, H \rangle - \alpha (H, H), \\ L_2 &= \psi_+^I (\nabla^2 \mathcal{F}_0)_{IJ} \psi_-^J + 2(\psi_+, C\eta_-) - 2(\psi_-, C\eta_+), \\ L_3 &= (\phi_{+-}, C^\dagger C \phi_{+-}) - 2(\phi_{+-}, K(\psi_-, \psi_+)), \\ L_4 &= -(\phi_{--}, C^\dagger C \phi_{++} - K(\psi_+, \psi_+)) + (\phi_{++}, K(\psi_-, \psi_-)). \end{aligned} \quad (5.4)$$

The four terms of the Lagrangian play distinguished roles in the evaluation of the path integral, and can be discussed separately:

- $L_1$  is the familiar localization to the critical points of the action potential. The evaluation of the path integral near these critical points gives

$$\frac{1}{|\text{Det}' \nabla^2 \mathcal{F}_0|}. \quad (5.5)$$

- $L_2$  is the fermion Lagrangian associated with the deformation complex (4.36) and (4.39) of the equations  $\nabla \mathcal{F}_0 = 0$ . Note that gauge invariance of  $\mathcal{F}_0$  guarantees that this is a complex since  $\nabla^2 \mathcal{F} C \eta = 0$  at the critical points.

Note that the virtual dimension of the moduli space is automatically zero: in the balanced theory there are as many ghost zero modes as antighost zero modes, and they live in the same bundle. The fermion operator is thus:

$$\begin{pmatrix} \nabla^2 \mathcal{F}_0 & C \\ C^\dagger & 0 \end{pmatrix}. \quad (5.6)$$

Because we assume that  $\mathcal{F}_0$  is a nondegenerate Morse function we can block diagonalize into the kernel of  $\nabla^2 \mathcal{F}_0$  and its orthogonal subspace:

$$\begin{pmatrix} (\nabla^2 \mathcal{F}_0)' & 0 & 0 \\ 0 & 0 & C \\ 0 & C^\dagger & 0 \end{pmatrix} \quad (5.7)$$

There is also a finite-dimensional space of fermion zero modes associated to the tangent to moduli space, or, better, to the cohomology of the complex (4.36).

The determinant of the fermion non-zero modes is

$$\det'(\nabla^2 \mathcal{F}_0) \cdot \det(C^\dagger C). \quad (5.8)$$

•  $L_3$ : The integral is gaussian, so that  $\phi_{+-}$  effectively localizes to zero. (More precisely, it localizes to an even nilpotent.) The path integral gives:

$$\frac{1}{\sqrt{\det C^\dagger C}} \exp \frac{1}{\alpha} \left[ (K(\psi_-, \psi_+), \frac{1}{C^\dagger C} K(\psi_-, \psi_+)) \right]. \quad (5.9)$$

•  $L_4$ : This is also a gaussian integral and gives:

$$\frac{1}{\det C^\dagger C} \exp \frac{1}{\alpha} \left[ (K(\psi_+, \psi_+), \frac{1}{C^\dagger C} K(\psi_-, \psi_-)) \right]. \quad (5.10)$$

Notice that the determinants of  $C$ 's do not cancel. The reason is that we have not fixed the gauge. This can be very elegantly solved using the differential topology that we introduced in section 3. We can include naturally the ghosts *as well as the antighosts* of  $G$ -gauge fixing by passing to the Weil model (instead of the Cartan model) of equivariant cohomology, and introducing a gauge-noninvariant term in the action potential.

Recall that in the case  $N_T = 2$  the Weil multiplet consists of a triplet  $(\omega_+, \omega_-, \Omega)$  of connections, see (3.16). Here the connection  $\omega_+$  appears as the ghost. The connections  $(\omega_-, \Omega)$  represent the antighost multiplet. The gauge fixing Lagrangian is written as

$$\begin{aligned} d_+ d_- (\epsilon u^2) &= \epsilon d_+ (\omega_-, C^\dagger u) + \epsilon d_+ (u, \psi_-) \\ &= \epsilon (\Omega, C^\dagger u) + \epsilon (\omega_-, C^\dagger C \omega_+) + \epsilon d_+ (u, \psi_-). \end{aligned} \quad (5.11)$$

The integrals over the first two terms provide the missing  $\sqrt{\det C^\dagger C}$ . The last term adds some gauge-noninvariant pieces to the “matter” Lagrangian, but we can invoke  $\epsilon$ -independence to argue that these terms make no contribution.

The net result of the path integral is an integral over collective coordinates:

$$\int_{\mathcal{M}} \prod du_0^I d\psi_+^{0,I} d\psi_-^{0,I} \exp\left((K(\psi_+, \psi_+), \frac{1}{C^\dagger C} K(\psi_-, \psi_-))\right). \quad (5.12)$$

Finally, let us recall that if  $E_{1,2}$  are trivial hermitian vector bundles and  $A$  is a linear fiber map  $A : E_2 \rightarrow E_1$  then there is a natural connection on  $\ker A^\dagger \subset E_1$  given by  $P \circ d$  where  $P$  is the projection operator. The curvature is just

$$R = PdA \frac{1}{A^\dagger A} dA^\dagger P \quad (5.13)$$

In our case  $C : \mathfrak{g} \rightarrow T\mathcal{C}$  and the tangent bundle to the moduli space is  $T\mathcal{M} \cong \ker C^\dagger$ . We recognize this form in the remaining integral (5.12). Putting all this together we obtain the result (4.24).

### 5.1. Localization of the cofield model: “counting without signs”

The cofield model can be put into the standard framework by taking the field space to be  $E^* \rightarrow M$  and the antighost bundle to be  $\pi^*(E \oplus T^*M) \rightarrow E^*$ . We choose the section  $\mathbf{s} = (s, \nabla_\mu s^a \hat{u}_a)$  and localize to (4.31). For simplicity suppose  $\nabla_\mu s^a$  has no kernel and the index is all cokernel. Then we localize to  $\hat{u} = 0$ . Note that the fermionic operator is

$$\nabla^2 \mathcal{F}_0 = \begin{pmatrix} 0 & \nabla_\mu s^a \\ (\nabla_\mu s^a)^\dagger & 0 \end{pmatrix} \quad (5.14)$$

For this reason the fermionic path integral is always positive semidefinite and we are “counting without signs” [4]. In any case, the result is:  $Z = \chi(\mathcal{Z}(s))$ , which was, of course, the original motivation for the cofield construction [1].

## 6. Examples of BTFT’s

In this section we briefly mention some important examples of balanced topological field theories in various dimensions. Note that in principle we have a map that associates to any local QFT action  $\mathcal{F}$  a BTFT, by simply using  $\mathcal{F}$  as action potential. Of course, to get a reasonable action for the BTFT, for example quadratic in derivatives, the action potential should satisfy certain constraints. Typically it will be first order in derivatives. Fortunately, there are quite a few interesting candidates of that form.

### 6.1. Morse theory

Take  $X$  to be a finite dimensional Riemannian manifold and  $\mathcal{F}_0$  to be a Morse function. This is the standard example to which supersymmetric quantum mechanics on  $X$  ( $SMQ(X)$ ) reduces. The path integral becomes:

$$Z = \int \exp \left[ -iH^\mu \nabla_\mu \mathcal{F}_0 - \epsilon G_{\mu\nu} H^\mu H^\nu + \epsilon[(\psi_+, \nabla^2 \mathcal{F}_0 \psi_-) + \dots] \right] \quad (6.1)$$

where the ellipses indicate various curvature terms. Note that if  $\mathcal{F}_0 = \frac{1}{2} \sum \lambda_i u_i^2$  then the quadratic term in the Lagrangian is  $(\nabla \mathcal{F}_0)^2 = \sum \lambda_i^2 u_i^2$ . This is indeed the canonical example. If we choose  $U = (u, \psi_\pm, H) \in \widehat{\mathbb{R}}^n = \mathbb{R}^{2n|2n}$  with  $\mathcal{F}(U) = \frac{i}{2} u A u + \psi_+ B \psi_-$  and  $A, B$  quadratic forms, the fundamental gaussian integral is

$$\frac{1}{(2\pi i)^n} \int_{\widehat{\mathbb{R}}^n} \exp d_+ d_- \mathcal{F} = \text{sign}(\det A). \quad (6.2)$$

The determinants cancel, and the result does not depend on the choice of  $A$  and  $B$ , up to a sign. So we see that  $Z$  reduces to the sum of the indices of the critical points, and indeed equals the Euler number  $\chi(X)$ .

### 6.2. Balanced quantum mechanics

Ironically, one cannot obtain  $SQM(X)$  as a balanced theory, in spite of the fact that  $Z = \chi(X)$  for  $SQM(X)$  [12]. The balanced theory must necessarily have an action of the form:

$$\int_{S^1} dt \omega_{\mu\nu} [\dot{x}^\mu H^\nu - \dot{\psi}_+^\mu \psi_-^\nu - \epsilon H^\mu H^\nu], \quad (6.3)$$

where  $\omega = \omega_{\mu\nu} dx^\mu dx^\nu$  is a closed two-form.

A very natural class of such theories is provided by a symplectic target space  $(X, \omega)$ . Our field space is in that case  $LX$ , the space of closed unbased loops. The action potential leading to (6.3) is just

$$\mathcal{F}_0 = \oint_{S^1} \alpha_\mu \dot{x}^\mu = \int_D x^* \omega, \quad (6.4)$$

where  $d\alpha = \omega$  and we consider the circle to be the boundary of a disk  $D$ . Moreover, if  $H(x(t), t)$  is a time-dependent Hamiltonian then it is natural to consider the more general action potentials:

$$\mathcal{F}_0 = \int_D x^* \omega + \oint H(x(t), t) dt \quad (6.5)$$

Morse theory based on this functional is the subject of symplectic Floer homology [13].

### 6.3. Balanced $\sigma$ -models

There are many natural action potentials one might want to consider in the context of sigma-models. For example  $\mathcal{F}_0 = \int (\nabla f)^2$  would lead to a theory which calculates the Euler character of the moduli space of harmonic maps. Closely related actions have appeared in [14][15]. Other obvious choices are the Nambu action  $\mathcal{F}_0 = \text{Area}(f(\Sigma))$ . Such actions lead to nonrenormalizable actions. For example, the harmonic map choice leads to an action fourth-order in derivatives. (N.B. The theory is easily generalized to four dimensions). For this reason we focus on a particular case, described in the next section.

### 6.4. Cofield $\sigma$ -models

We describe the cofield construction for topological sigma models [1][2]. Begin with the standard moduli problem from holomorphic maps:  $E \rightarrow \text{MAP}(\Sigma, X)$ . The fields in  $\widehat{E}$  fit into two quartets:

$$\begin{array}{ccccc}
 & & \psi^i & & \\
 & \nearrow & & \searrow & \\
 x^i & & & & \bar{H}^i \\
 & \searrow & & \nearrow & \\
 & & \bar{\pi}^i & & 
 \end{array}
 \quad
 \begin{array}{ccccc}
 & & \pi^{\bar{z}}_i & & \\
 & \nearrow & & \searrow & \\
 p^{\bar{z}}_i & & & & \bar{H}^{\bar{z}}_i \\
 & \searrow & & \nearrow & \\
 & & \bar{\psi}^{\bar{z}}_i & & 
 \end{array}
 \quad (6.6)$$

and

$$\begin{array}{ccccc}
 & & \bar{\psi}^{\bar{i}} & & \\
 & \nearrow & & \searrow & \\
 \bar{x}^{\bar{i}} & & & & H^{\bar{i}} \\
 & \searrow & & \nearrow & \\
 & & \pi^{\bar{i}} & & 
 \end{array}
 \quad
 \begin{array}{ccccc}
 & & \bar{\pi}^{z_{\bar{i}}} & & \\
 & \nearrow & & \searrow & \\
 \bar{p}^{z_{\bar{i}}} & & & & H^{z_{\bar{i}}} \\
 & \searrow & & \nearrow & \\
 & & \psi^{z_{\bar{i}}} & & 
 \end{array}
 \quad (6.7)$$

where  $i, \bar{i}$  are holomorphic (anti-holomorphic) indices on the target space  $X$  and  $z, \bar{z}$  are holomorphic (anti-holomorphic) on the worldsheet. As we will discuss in section 7 these fields will describe a conformal field theory.

The action potential is:

$$\mathcal{F}^{BT\sigma} = i\mathcal{F}_0 + \mathcal{F}^{metric}, \quad (6.8)$$

where

$$\begin{aligned}
 \mathcal{F}_0 &= \int_{\Sigma} \sqrt{h} \left[ p^{\bar{z}}_i \bar{\partial}_{\bar{z}} x^i + \bar{p}^{z_{\bar{i}}} \partial_z \bar{x}^{\bar{i}} \right], \\
 \mathcal{F}^{metric} &= \int_{\Sigma} \sqrt{h} h^{z\bar{z}} \left[ (\bar{\psi}^{\bar{z}}_i \quad -\bar{\pi}^i) \mathbf{L} \begin{pmatrix} \bar{\pi}^{z_{\bar{j}}} \\ \bar{\psi}^{\bar{j}} \end{pmatrix} + (\psi^{z_{\bar{j}}} \quad -\pi^{\bar{j}}) \mathbf{L}^{Tr} \begin{pmatrix} \pi^{\bar{z}}_i \\ \psi^i \end{pmatrix} \right],
 \end{aligned}
 \quad (6.9)$$

where  $h$  is a metric on  $\Sigma$  and  $\mathbf{L}$  is a metric related to the hyperkähler metric on  $T^*X$ , as described in section 7 below.

If there is a moduli space but no antighost zero modes, *i.e.* if  $\dim \text{cok } D_{\bar{z}} = 0, \dim \ker D_{\bar{z}} > 0$  where  $D_{\bar{z}} = \bar{\partial}_{T^*X} : \Omega^{0,0}(\Sigma; T^*X) \rightarrow \Omega^{0,1}(\Sigma; T^*X)$ , then we localize to  $\hat{f}_\mu^\alpha = 0$ ,  $f \in \text{HOL}(\Sigma, X)$ , the space of holomorphic maps. Furthermore, the path integral is given by  $Z = \chi(\text{HOL}(\Sigma, X))$ . This situation is uncommon.

### 6.5. Balanced topological 2D gravity: Beltrami formulation

There is also a balanced version of topological gravity. We can give two (equivalent) definitions, either using the language of metrics or of complex curves. We start with the latter point of view. In that case the relevant moduli problem is a pair  $(C, V)$  with  $C$  a complex curve of genus  $g$  and  $V$  a holomorphic vector field on  $C$ . Since for  $g > 1$  such a vector field is generically zero, the moduli space reduces to the moduli space  $\mathcal{M}_g$  of curves. However, the virtual dimension of the moduli problem is zero and the theory is thus balanced. So, by definition this model computes  $\chi(\mathcal{M}_g)$ .

In more detail: We fix a complex structure and consider the Beltrami differentials  $\mu_{\bar{z}}^z \in \mathcal{B}^{(-1,1)}$  which modify the Dolbeault operator to

$$\bar{\partial}^{(\mu)} = \bar{\partial}_{\bar{z}} + \mu \partial_z. \quad (6.10)$$

The deformation complex becomes:

$$0 \longrightarrow \text{Vect}^{1,0} \xrightarrow{C} \mathcal{B}^{(-1,1)} \oplus \text{Vect}^{1,0} \xrightarrow{D} \mathcal{B}^{(-1,1)} \longrightarrow 0, \quad (6.11)$$

with

$$C\eta_- = \begin{pmatrix} \bar{\partial}^{(\mu)}\eta_- \\ [\hat{f}, \eta_-] \end{pmatrix}, \quad D(\mu, \eta_+) = ([\hat{f}, \mu], \bar{\partial}^{(\mu)}\eta_+), \quad (6.12)$$

where the “cofield”  $\hat{f}$  is a vector field, and one must take care to write:

$$\begin{aligned} (\bar{\partial}^{(\mu)}V)^z \frac{\partial}{\partial z} &\equiv \left[ (\bar{\partial}_{\bar{z}} + \mu \partial_z)V^z + [\mu, V]^z \right] \frac{\partial}{\partial z}, \\ [\mu, V]^z &= \mu \partial_z V^z - V^z \partial_z \mu. \end{aligned} \quad (6.13)$$

As we mentioned above, interpreted as an ordinary topological field theory we have the moduli problem of a holomorphic vector field and a complex curve  $(C, V)$ . For  $g > 1$  there are no nonsingular holomorphic vector fields. Thus, the localization to  $\hat{f} = 0$  makes sense and the path integral computes the orbifold Euler character of moduli space.

### 6.6. Balanced topological 2D gravity: metric formulation

An alternative formulation of balanced topological gravity starts from metrics. Let  $MET$  denote the space of Riemannian metrics  $h_{\alpha\beta}$  on a topological surface  $\Sigma$ . The basic quartet of  $\widehat{\widehat{MET}}$  will be denoted as

$$\begin{array}{ccc}
 & \psi_{\alpha\beta,+} & \\
 \nearrow & & \searrow \\
 h_{\alpha\beta} & & k_{\alpha\beta} \\
 \searrow & & \nearrow \\
 & \psi_{\alpha\beta,-} &
 \end{array} \tag{6.14}$$

We continue to take  $\text{Diff}(\Sigma)$  as the gauge group, since there are no Weyl-invariant metrics on  $MET$ . A natural choice of action is

$$I = d_+ d_- \left( \int \sqrt{h} h^{\alpha\beta} k_{\alpha\beta} \right), \tag{6.15}$$

but an equivalent and more convenient choice of action potential is:

$$\begin{aligned}
 \mathcal{F}^{BTG} &= (\psi_+, \psi_-), \\
 d_- \mathcal{F}^{BTG} &= (H, \psi_-) + (\mathcal{R}_{-+} u, \psi_-) - (\mathcal{R}_{--} u, \psi_+) \\
 &= \int \sqrt{h} \left[ \rho_{\bar{z}}^z (D_z \hat{f}^{\bar{z}} - H_{\bar{z}}^{\bar{z}}) + c.c. \right] + \int \sqrt{h} \lambda^\alpha \nabla^\beta \psi_{\alpha\beta}.
 \end{aligned} \tag{6.16}$$

Translating the fields to the standard notation for 2D gravity (see, e.g. [2], sec. 16.2) we have:

$$u_{\alpha\beta} \rightarrow \delta h_{\alpha\beta}, \quad \psi_- \rightarrow \rho, \quad \phi_{+-} \rightarrow \hat{f}^\alpha, \quad \phi_{--} \rightarrow \lambda^\alpha, \quad \phi_{++} \rightarrow \gamma^\alpha. \tag{6.17}$$

Again we recognize the gauge fermion appropriate to the moduli problem of a pair  $(C, V)$ ,  $C$  a curve and  $V$  a holomorphic vector field. <sup>4</sup>

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<sup>4</sup> We still must choose an action potential that fixes the Weyl mode. In principle any  $\text{Diff}(\Sigma)$ -invariant functional of the metric  $\mathcal{F}_0[h_{\alpha\beta}]$  which takes a *unique* minimum in each conformal class can serve as  $\mathcal{F}_0$ .

### 6.7. Balanced topological strings

The coupling of the sigma model to gravity is simply summarized by taking the sum of the action potentials  $\mathcal{F} = \mathcal{F}^{BTG} + \mathcal{F}^{BT\sigma}$  and using the Diff-equivariant version of  $d_A$ . This completely encodes the coupling to balanced topological gravity ! Let us study the resulting coupling to gravity.

We separate the BRST operator into the part that varies the graviton and the rest:

$$d_A = d_A^0 + \psi_{A,\alpha\beta} \frac{\delta}{\delta g_{\alpha\beta}}. \quad (6.18)$$

The action may then be expressed as

$$\begin{aligned} d_+ d_- \mathcal{F} &= d_+^0 d_-^0 \mathcal{F} + \epsilon^{AB} d_A^0 \left( \frac{\delta \mathcal{F}}{\delta g_{\alpha\beta}} \right) \psi_{B,\alpha\beta} \\ &+ \left( \frac{\delta \mathcal{F}}{\delta g_{\alpha\beta}} \right) (k_{\alpha\beta} + \dots) + \left( \frac{\delta^2 \mathcal{F}}{\delta g_{\alpha\beta} \delta g_{\gamma\delta}} \right) \psi_{+,\alpha\beta} \psi_{-,\gamma\delta} \end{aligned} \quad (6.19)$$

Thus, the auxiliary field  $k_{\alpha\beta}$  couples to the stress tensor  $K_{\alpha\beta}$  of the action potential, while the two partners  $\psi_{A,\alpha\beta}$  of the graviton couple to the variations of the gauge fermions:

$$G_{A,\alpha\beta} \equiv d_A^0 \left( \frac{\delta \mathcal{F}}{\delta g_{\alpha\beta}} \right) = \frac{\delta \Psi_A^0}{\delta g_{\alpha\beta}}. \quad (6.20)$$

The four currents  $K_{\alpha\beta}, G_{A,\alpha\beta}, T_{\alpha\beta}$  fit into a quartet:

$$\begin{array}{ccc} & G_{+,\alpha\beta} & \\ K_{\alpha\beta} & \nearrow & \searrow T_{\alpha\beta} \\ & G_{-,\alpha\beta} & \nearrow \end{array} \quad (6.21)$$

Specifically, for the cofield sigma model:  $K_{zz} = \pi_{zi} \partial x^i + \dots$ ,  $K_{\bar{z}\bar{z}} = \bar{\pi}_{\bar{z}\bar{i}} \bar{\partial} \bar{x}^{\bar{i}} + \dots$

### 6.8. 2D Yang-Mills

There is an obvious choice for an action potential that is first order in derivatives for a two-dimensional gauge theories. Consider a connection  $A$  together with a Lie-algebra valued scalar field  $\phi$  on a Riemann surface  $\Sigma$ . Choose the action potential

$$\mathcal{F} = \int \text{Tr}(\phi F), \quad (6.22)$$

which has its critical points on the moduli space of flat connections on  $\Sigma$ . The resulting action will be of the form  $I = \int F_{\mu\nu}^2 + \dots$ . In fact, this model is rather familiar, since it corresponds directly to the reduction to two dimensions of four-dimensional Donaldson theory [16].



### 6.9. 3D Chern-Simons

For a three-dimensional gauge theory there is also a canonical choice for a first-order action potential: the famous Chern-Simons term. Note that quite generally, for any gauge theory we have the field quartet

$$\begin{array}{ccccc}
 & & \psi_{+,\mu} & & \\
 & \nearrow & & \searrow & \\
 A_\mu & & & & H_\mu \\
 & \searrow & & \nearrow & \\
 & & \psi_{-,\mu} & & 
 \end{array} \tag{6.23}$$

while the Cartan multiplet  $\phi_{AB}, \eta_A$  are  $\mathfrak{g}$ -valued fields on spacetime  $X$ .

If we choose the three-dimensional action potential

$$\mathcal{F} = \int_X \text{Tr}(AdA + \frac{2}{3}A^3) + \text{Tr} \psi_+ \psi_-, \tag{6.24}$$

the resulting action is, according to our general formulae:

$$\begin{aligned}
 I = \int_X \text{Tr} \Big[ & FH - \epsilon H^2 + 2(D\eta_+ \psi_- - D\eta_- \psi_+) \\
 & + \psi_+ [\phi_{--}, \psi_+] + \psi_- [\phi_{++}, \psi_-] - 2\psi_- [\phi_{+-}, \psi_+] \\
 & + (D\phi_{+-})^2 - D\phi_{++} D\phi_{--} \Big] \tag{6.25}
 \end{aligned}$$

This turns out to be the reduction to three dimensions of Donaldson theory. The Morse theory problem in this case defines Floer's 3-manifold homology theory. The theory computes the Euler number of the moduli space of flat connections on  $X$ . One can similarly discuss the *IG* theories of [17]. See also the work of [5].

### 6.10. 4D Yang-Mills

This is the context of the twisting of  $N = 4$  supersymmetric Yang-Mills discussed in [4]. We now consider the cofield construction applied to Donaldson theory. This should calculate the Euler character of the moduli space of self-dual instantons. According to the cofield construction we should have two quartets:

$$\begin{array}{ccccccc}
 & & \psi_{+,\mu} & & & & \psi_{+,\mu\nu} \\
 & \nearrow & & \searrow & & \nearrow & \\
 \text{fields } A_\mu & & & & H_\mu & \text{equations : } B_{\mu\nu} & & H_{\mu\nu} \\
 & \searrow & & \nearrow & & \searrow & \nearrow & \\
 & & \psi_{-,\mu} & & & & \psi_{-,\mu\nu} & 
 \end{array} \tag{6.26}$$

In addition we have the Cartan quintet for YM gauge symmetry, as above.

The naive cofield construction would suggest the action potential  $\mathcal{F}_0 = \int_X BF^+$ , but to match with the twisted action of  $N = 4$  SYM one must take a modified action potential. The correct choice is

$$\begin{aligned}
\mathcal{F} &= \mathcal{F}_1 + \mathcal{F}_2 + \mathcal{F}_3 \\
\mathcal{F}_1 &= \int_X \text{Tr}(B^{\mu\nu} F_{\mu\nu}^+ + \frac{1}{12} B^{\mu\nu} [B_{\mu\lambda}, B_\nu^\lambda]) \\
\mathcal{F}_2 &= \int_X \text{Tr}(\psi_-^{\mu\nu} \psi_{+,\mu\nu} + \psi_-^\mu \psi_{+,\mu}) \\
\mathcal{F}_3 &= \int_X \text{Tr}(\eta_+ \eta_-)
\end{aligned} \tag{6.27}$$

The balanced 4D YM theory may be identified with a twist of the  $N = 4$  SYM theory as described in [4]. We embed  $SU(2)_R$  into the internal  $SU(4)$  symmetry of  $N = 4$  SYM so that  $2 + 2 = 4$ . The fermion multiplets then become:

$$\begin{aligned}
(2, 1, 4) \oplus (1, 2, \bar{4}) &= (2, 2) \oplus (2, 2) \oplus (1, 1) \oplus (1, 3) \oplus (1, 1) \oplus (1, 3) \\
&= (\psi_{+,\mu}) \oplus (\psi_{-,\mu}) \oplus (\eta_+) \oplus (\psi_{+,\mu\nu}) \oplus (\eta_-) \oplus (\psi_{-\mu\nu})
\end{aligned} \tag{6.28}$$

The unbroken internal  $SU(2)$  symmetry is the  $sl_2$  symmetry of the balanced theory. Similarly we obtain the scalars from  $\psi^{IJ} = (4 \times 4)_{\text{antisymm}} \rightarrow 3 \times (1, 1) + (1, 3)$  giving  $B_{\mu\nu}, \phi_{AB}$ . Adding  $\mathcal{F}_3$  makes the twisted theory closer to the physical theory, by giving a potential energy to the scalars.

## 7. A new twist on the topological sigma model

Just the way the balanced four-dimensional Yang-Mills theory is related by a twist to the  $N = 4$  supersymmetric Yang-Mills theory, the balanced topological string on a Kähler target space  $X$  is closely related to an  $N = 4$  string with target space  $T^*X$ . These strings are quite interesting, since the balancing property implies that they are critical in any dimension.<sup>5</sup>

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<sup>5</sup> The work in this section was done in collaboration with K. Intriligator and R. Plesser.

### 7.1. Free $N = 4$ Multiplets

As a warmup we consider a single free  $N = 4$  multiplet that we write as

$$\begin{aligned} X_{A\dot{B}} &= \begin{pmatrix} x & \bar{p} \\ -p & \bar{x} \end{pmatrix} \\ \psi_{A\dot{B}} &= \begin{pmatrix} \psi & \bar{\pi} \\ -\pi & \bar{\psi} \end{pmatrix} \end{aligned} \quad (7.1)$$

When we generalize to  $d$  such multiplets we should regard it as defining a hyperkähler sigma model with target space  $T^*\mathbb{C}^d$ . On shell, this theory has a large  $N = 4$  superconformal symmetry. We focus on the aspects that generalize to a more general target  $T^*X$  with  $X$  a Kähler manifold. The four supercurrents are

$$G_{\dot{A}\dot{B}} = \psi_{A\dot{A}} \partial X_{B\dot{B}} \epsilon^{AB} = \begin{pmatrix} \pi \partial x - \psi \partial p & \pi \partial \bar{p} + \psi \partial \bar{x} \\ -\bar{\psi} \partial x - \bar{\pi} \partial p & -\bar{\psi} \partial \bar{p} + \bar{\pi} \partial \bar{x} \end{pmatrix} \quad (7.2)$$

We furthermore have an  $SU(2)_L \times SU(2)_R$  current algebra. The right currents are given by

$$J_{\dot{A}\dot{B}} = \frac{1}{2} \psi_{A\dot{A}} \psi_{B\dot{B}} \epsilon^{AB}. \quad (7.3)$$

These three currents correspond to the three Kähler forms  $\omega_C, \omega_R, \omega_C^*$  in the case of a general hyperkähler manifold.

We will be interested in targets for which there is an additional  $U(1)$  isometry of the metric. In the present case the  $U(1)$  isometry current is:

$$\begin{aligned} J_z^{isom} &= \pi_{+,i} \bar{\pi}_{+,\bar{i}} - p_i \partial \bar{p}_{\bar{i}} \\ \tilde{J}_{\bar{z}}^{isom} &= \bar{\pi}_{-,\bar{i}} \pi_{-,i} - \bar{p}_{\bar{i}} \partial p_i \end{aligned} \quad (7.4)$$

Note that the isometry current is not a conformal current, and the conservation law is  $\bar{\partial} J_z - \partial \tilde{J}_{\bar{z}} = 0$ . Nevertheless, if one proceeds naively and evaluates the OPE's for on-shell fields, one finds that the charges of the fields under this current are:

$$\begin{aligned} J_z^{isom}(z) \cdot \pi(w) &\sim \frac{1}{z-w} \pi(w) \\ J_z^{isom}(z) \cdot p(w) &\sim \frac{1}{z-w} p(w) \\ J_z^{isom}(z) \cdot \partial \bar{p}(w) &\sim -\frac{1}{z-w} \partial \bar{p}(w) \end{aligned} \quad (7.5)$$

Now recall that the standard topological twist of an  $N = 4$  multiplet is defined as the following modification of the stress-tensor

$$\begin{aligned} T' &= T + \partial J_{+\dot{-}} \\ \tilde{T}' &= T - \bar{\partial} \tilde{J}_{+\dot{-}} \end{aligned} \tag{7.6}$$

This gives the standard A-model for  $T^*\mathbb{C}^d$  [18].

We now describe the new twist, which we call the “isometry twist” or “I-twist.” In terms of conformal field theory the isometry twisted model is related to the  $T^*\mathbb{C}^d$  A-model by the twists:

$$\begin{aligned} T'' &= T + \partial J_{+\dot{-}} - \partial J^{isom} = T' - \partial J^{isom} \\ \tilde{T}'' &= \tilde{T}' - \bar{\partial} \tilde{J}^{isom} \end{aligned} \tag{7.7}$$

The field content of the I-twisted model off shell is described by the bosonic fields  $x^i, \bar{x}^{\bar{i}}, p_i^{\bar{z}}, p_{\bar{i}}^z$ , the ghost number one fields  $\psi^i, \bar{\psi}^{\bar{i}}, \pi_i^{\bar{z}}, \bar{\pi}_{\bar{i}}^z$ , and the ghost number  $-1$  fields  $\bar{\psi}_{\bar{z}}^{\bar{i}}, \psi_z^i, \pi_i, \bar{\pi}_{\bar{i}}$ . On shell, we have holomorphic fields:  $p_{zi}, \bar{p}_{\bar{i}}^z, \psi^i, \bar{\psi}_{\bar{z}}^{\bar{i}}, \pi_{zi}, \bar{\pi}_{\bar{i}}$  and similarly for anti-holomorphic fields. In particular, the anti-holomorphic bosonic fields include  $\tilde{p}_{\bar{z}\bar{i}}, \tilde{p}_{\bar{i}}^{\bar{z}}, \tilde{\pi}_{\bar{z}\bar{i}}, \tilde{\pi}_{\bar{i}}$ . We summarize a comparison of dimensions for holomorphic conformal fields in the following table. Here  $\Delta'$  and  $\Delta''$  indicate the conformal dimensions in the usual A-twist and the new I-twist respectively.

| operator  | $\Delta'$ | $\Delta''$ |
|---|-----------|------------|
| $\psi$  | 0         | 0          |
| $\bar{\psi}$  | 1         | 1          |
| $\pi$   | 0         | 1          |
| $\bar{\pi}$   | 1         | 0          |
| $p$   | 0         | 1          |
| $\partial \bar{p}$  | 1         | 0          |
| $\pi \partial x - \psi \partial p$                          | 1         | 2          |
| $\pi \partial \bar{p} + \psi \partial \bar{x}$              | 1         | 1          |
| $-\bar{\psi} \partial x - \bar{\pi} \partial p$             | 2         | 2          |
| $-\bar{\psi} \partial \bar{p} + \bar{\pi} \partial \bar{x}$ | 2         | 1          |
| $J_{++}$  | 0         | 1          |
| $J_{+\dot{-}}$  | 1         | 1          |
| $J_{++}$  | 2         | 1          |

Note the unusual feature that in the I-twist a bosonic current gets twisted. This is one of the most interesting aspects of the isometry twist. Note also that the isometry current is BRST exact:

$$J^{isom} = \{ \oint \pi \partial \bar{p} + \psi \partial \bar{x}, -p \bar{\pi} \}. \quad (7.8)$$

Thus, even though it is not a good conformal current, the resulting model is well-defined.

The currents from the isometry twist couple to gravity as in (6.21) with:

$$\begin{aligned} K &= p \partial x, \\ G_+ &= \pi \partial x + p \partial \psi, \\ G_- &= \bar{\psi} \partial x - p \partial \bar{\pi}, \\ T &= \partial \bar{x} \partial x + p \partial (\partial \bar{p}) + \pi \partial \bar{\pi} + \bar{\psi} \partial \psi. \end{aligned} \quad (7.9)$$

## 7.2. Hyperkähler metric on $T^*X$

Suppose  $X$  is a Kähler manifold with metric  $G_{i\bar{j}} dx^i d\bar{x}^{\bar{j}}$  and corresponding Kähler form  $\omega$ . Let  $K_0$  be the Kähler potential. The noncompact manifold  $T^*X$  has a hyperkähler metric (of signature  $(n, n)$ )  $\mathbb{G}$  on  $T^*X$  [19]. To make this plausible note that  $c_1(T^*X) = 0$  and that, in terms of local holomorphic coordinates  $(z^i, p_i)$  on  $T^*X$ , there is a very natural nonvanishing holomorphic 2-form:  $\omega_C = dz^i \wedge dp_i$ . We denote the components of this hyperkähler metric on  $T^*X$  as:

$$ds^2 = \mathbb{G}_{i\bar{j}} dx^i d\bar{x}^{\bar{j}} + \mathbb{G}_{\bar{j}}^i Dp_i d\bar{x}^{\bar{j}} + \mathbb{G}_i^{\bar{j}} dx^i D\bar{p}_{\bar{j}} + \mathbb{G}^{i\bar{j}} Dp_i D\bar{p}_{\bar{j}} \quad (7.10)$$

The Kähler potential is of the form

$$K = f(\xi), \quad \xi = G^{i\bar{j}}(x) p_i \bar{p}_{\bar{j}} = \|p\|^2 \quad (7.11)$$

and hence the metric has the required  $U(1)$  isometry in the tangent directions.

*Example.* One example of this construction has appeared in the theory of the  $N = 2$  string [20]. Let  $X$  be the upper half plane with Poincare metric and  $\xi = (Imz)^2 \|w\|^2$ . Then the Kähler potential for the hyperkähler metric is:<sup>6</sup>

$$K = 2\sqrt{c\xi + e^2} + e \log \left[ \frac{\sqrt{c\xi + e^2} - e}{\sqrt{c\xi + e^2} + e} \right] \quad (7.12)$$

To avoid a singularity we must take  $c > 0$ . The construction is  $SL(2, \mathbb{R})$  invariant and thus defines a hyperkähler metric on the cotangent bundles to Riemann surfaces of genus  $g > 1$ .

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<sup>6</sup> Actually, the signature is  $(2, 2)$  so the metric is *hypersymplectic*. See [21] for a careful discussion of the signs involved.

### 7.3. Isometry twisted $\sigma$ -model in the general case

Using the isometry we twist in the manner described above. In order to do this with the sigma model action one must first add a topological Kähler term to make the bosonic part of the action chiral. The momentum coordinates pick up conformal spins  $\pm 1$ . Consequently the off-diagonal parts of the metric  $\mathbb{G}$  obtain conformal spin.

We then proceed as follows. Define

$$\mathbb{G} = \begin{pmatrix} h^{z\bar{z}}\mathbb{G}_{i\bar{j}} & \mathbb{G}_{\bar{j}}^{zi} \\ \mathbb{G}_i^{\bar{z}\bar{j}} & \mathbb{G}^{i\bar{j}} \end{pmatrix} \quad (7.13)$$

The gauge fermion of the I-twisted model will be:

$$\Psi_- = \int \sqrt{h} \left[ (\bar{\psi}_{\bar{z}}^{\bar{j}} \quad \bar{\pi}_{\bar{j}}) \mathbb{G} \begin{pmatrix} \bar{\partial}x^i - H_{\bar{z}}^i \\ D_{\bar{z}}p_i^{\bar{z}} - H_i \end{pmatrix} + (\psi_{\bar{z}}^i \quad \pi_i) \mathbb{G}^{tr} \begin{pmatrix} \partial x^{\bar{j}} - H_{\bar{z}}^{\bar{j}} \\ D_z p_{\bar{j}}^z - H_{\bar{j}} \end{pmatrix} \right] \quad (7.14)$$

To relate this model to the balanced  $\sigma$  model we must relate the fields. We take:

$$\begin{aligned} \begin{pmatrix} \psi_{\bar{j}}^z \\ -\pi_{\bar{j}} \end{pmatrix} &= \mathbb{G} \begin{pmatrix} \psi_{\bar{z}}^i \\ \pi_i \end{pmatrix} \\ \begin{pmatrix} H_{\bar{j}}^z \\ H_{\bar{j}} \end{pmatrix} &= \mathbb{G} \begin{pmatrix} H_{\bar{z}}^i \\ H_i \end{pmatrix} \\ \begin{pmatrix} \bar{\psi}_{\bar{i}}^{\bar{z}} \\ -\bar{\pi}^i \end{pmatrix} &= \mathbb{G}^{tr} \begin{pmatrix} \bar{\psi}_{\bar{z}}^{\bar{j}} \\ \bar{\pi}_{\bar{j}} \end{pmatrix} \end{aligned} \quad (7.15)$$

and  $\mathbf{L} = \mathbb{G}^{-1}$ .

In order to relate this theory to the actual action written in [1][2] we need to use that for  $\xi \rightarrow 0$  the hyperkähler potential has the form:

$$K \rightarrow K_0 + a\xi + \mathcal{O}(\xi^2) + F(z^i) + F(z^i)^* \quad (7.16)$$

so that near  $\xi = 0$  the metric becomes a product metric. Since the theory localizes to  $\xi = 0$  (thus effectively killing half the bosonic degrees of freedom) the theories are effectively the same.

### 7.4. Relation to the $N = 2$ String

The matter systems described above appear in the  $N = 2$  string. Indeed, the twisting of the  $N = 4$  theory was used in [22] to produce topological field theory formulae for certain  $N = 2$  string amplitudes. However, the gravitational sector of the  $N = 2$  string and the balanced topological string appears to be different.

The string theory of large  $N$  2D Yang-Mills theory is a balanced topological string, and that lead to a conjecture that the balanced topological string for 4D balanced topological string is related to the large  $N$  limit of the Donaldson invariants [23][2]. A slightly different conjecture has been put forward in [24] relating the  $N = 2$  string to the large  $N$  limit of “holomorphic Yang-Mills” [25]. A better understanding of relation of the gravitational sector of balanced topological gravity and the gravitational sector of the  $N = 2$  string might shed some light on the compatibility of these conjectures, and even on the nature of 4D topological gauge theories.

## 8. Concluding Remarks

Some aspects of the above discussion deserve further investigation. For example, the isometry twist provides a novel method of eliminating bosonic zero modes, and thus provides a novel means of dimensional reduction. Also, there are subtle issues related to the fact that the current used in the twist is *not* a conformal current.

Naively, the absence of interesting cohomology on  $\widehat{X}$  suggests that there are no interesting observables. Moreover, the cancellation of the anomaly reinforces this. However, this is probably too naive since the action is itself  $d_+d_-$  exact and yet the path integral is not zero. This point remains to be clarified. The fact that balanced topological strings exist in any dimension is quite curious. In view of this it would be exciting to introduce observables into the theory.

Recently there has been intense study of “Dirichlet branes” or D-branes [26]. BPS states associated to D-branes are counted by Euler characters of certain moduli spaces. It would be interesting to see if one can apply BTFT’s to the study of D-branes.

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